

AUTOMORPHIC COMMUTATOR RELATIONS IN RELATIVE GEOMETRIC LANGLANDS

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ABSTRACT. We prove the automorphic commutator relation conjectured by Liu and Wang, which is the geometric input for the automorphic Clifford relations in their framework relating higher period integrals to higher derivatives of L -functions over function fields. The relation was previously established only under additional cohomological vanishing assumptions and for curves of genus different from one; we remove these assumptions. The main new ingredient is a “universal-local” stage between the local and the semi-local settings: we introduce the classifying stack of formal multidisks with two ordered marked sections, and we construct universal-local analogues of the Hecke stacks, unit objects, Hecke actions and special cohomological correspondences. Fusion and purity on this stack yield a universal-local commutator identity, which is proved using only the local assumptions on the Plancherel algebra that enter the conjecture. Specialization to a curve and the six-functor formalism then give the conjecture for every smooth projective curve.

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1. INTRODUCTION

1.1. Background and motivation.

1.1.1. *Special cycles and higher derivatives of L -functions.* L -functions are of central importance in number theory and Langlands program. Over function fields, higher derivatives of L -functions are governed by special cycles on moduli stacks of shtukas. This phenomenon was discovered by Yun and Zhang in their work on the higher Gross–Zagier formula [YZ17; YZ19]: intersection numbers of Heegner–Drinfeld cycles on the moduli stack of shtukas with r legs compute the r -th central derivative of the relevant L -function. Several further instances are now known. The higher Siegel–Weil formula of Feng, Yun and Zhang [FYZ24] identifies degrees of special cycles on moduli stacks of unitary shtukas with higher central derivatives of the non-singular Fourier coefficients of Siegel–Eisenstein series, and Feng, Howard and Mkrtchyan [FHM25] extended it to

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the coefficients of corank one. The arithmetic volume formulas of Feng, Yun and Zhang [FYZ26a] relate tautological classes on moduli stacks of shtukas to higher derivatives of Artin L -functions.

The same special-cycle classes appear as the coefficients of the higher theta series introduced in [FYZ21], which places their intersection theory within the Kudla program; the modularity results of [FYZ23; FK24; FYZ26b] lay the foundation for a higher arithmetic theta lift.

1.1.2. Relative Langlands duality. The relative Langlands duality of Ben-Zvi, Sakellaridis and Venkatesh [BSV24] provides a conjectural framework in which automorphic periods are related to L -functions through a duality between Hamiltonian spaces for a reductive group and for its dual group. Liu and Wang [LW26] and Wang [Wan25] extended this framework, in the strongly tempered setting, to higher derivatives and to intersection formulas on moduli stacks of shtukas. Their approach is sheaf-theoretic and rests on trace constructions, and its central observation is that higher-derivative formulas follow from automorphic Clifford relations satisfied by certain Hecke operators.

This framework gives a conceptual and uniform explanation of the results recalled in §1.1.1. In particular, [LW26; Wan25] establish intersection formulas for higher derivatives of Rankin–Selberg L -functions. Subsequent developments include the diagonal-cycle formulas for adjoint L -functions of Wang [Wan26] and the arithmetic volume formulas of Wang and Wei [WW26] for arbitrary coweights of split semisimple groups, which extend the results of [FYZ26a] and [Fen26].

1.1.3. Automorphic commutator relations. The key geometric input for the automorphic Clifford relations is the automorphic commutator relation, formulated in [LW26, Conjecture 4.45]. It was proved in [LW26, Theorem 4.47] under the additional Assumption 4.46 of *loc. cit.*, which requires three auxiliary cohomological vanishing statements and that the curve C have genus $g(C) \neq 1$. The main result of this paper, Main Theorem A, establishes the conjecture without any of these additional conditions.

As a consequence, the commutator identities used in the applications above now have a uniform proof. Specializing Main Theorem A to $X = \text{pt}$ with one representation trivial recovers [WW26, Theorems 3.2 and 3.8], the key ingredients in the proof of the main theorem [WW26, Theorem 2.13], which had to be proved separately because the hypotheses of [LW26, Theorem 4.47] are not satisfied there. It also removes the genus restriction from the global commutator identity of [Wan26, Proposition 2.6].

1.2. Notations and Settings. We use the conventions of [LW26, §2 and §4].

1.2.1. Group. Let G be a connected split reductive group over $\mathbb{F}_q$.

1.2.2. G -variety. Let $X = \text{Spec} A$ be an affine smooth G -variety together with a closed embedding $X \subset V$, where V is a finite-dimensional representation of G regarded as a G -variety.

1.2.3. Curve. Let C be a smooth projective geometrically connected curve over $\mathbb{F}_q$.

1.2.4. Coefficient field. Let k be the characteristic-zero ℓ -adic coefficient field, where $\ell \nmid q$, and let $\check{G}$ be the Langlands dual group over k .

1.2.5. Shift. For an integer n , we use the shift convention $\langle n \rangle = n$ for cohomological shift with half Tate twist.

1.2.6. Period sheaf. Let $\mathcal{P}_X = \pi_! k_{\text{Bun}_G^X}$ be the period sheaf, where $\pi : \text{Bun}_G^X \rightarrow \text{Bun}_G$, and Bun_G^X classifies a G -bundle on C together with a section of its associated X -bundle.

1.2.7. Representations. We use $I = [r]$ to denote an ordered set of cardinality $|I| = r$. Let V^i be a finite-dimensional representation of $\check{G}$ for $i \in [r]$. We set $V^I = \boxtimes_{i \in I} V^i$.

1.2.8. *Global Hecke stack and Hecke action.* We denote by $\mathrm{Hk}_{G,I}^{\mathrm{glob}}$ the global Hecke stack, whose S -points are

$$\mathrm{Hk}_{G,I}^{\mathrm{glob}}(S) = \{(c_I, \mathcal{E}_0, \mathcal{E}_1, \phi) \mid c_I \in C^I(S), \mathcal{E}_0, \mathcal{E}_1 \in \mathrm{Bun}_G(S), \\ \phi : \mathcal{E}_0|_{C_S - \Gamma_{c_i}} \xrightarrow{\cong} \mathcal{E}_1|_{C_S - \Gamma_{c_i}}\}.$$

We have

$$\mathrm{Corr}_{\mathrm{Hk}_{G,I}^{\mathrm{glob}}, \mathrm{IC}_{V^I}}(\mathcal{P}_X, \mathcal{P}_X\langle d_I - 2|I|\rangle) \cong \mathrm{Hom}^0(\mathrm{IC}_{V^I} \boxtimes \mathcal{P}_X, \mathcal{P}_X\langle d_I - 2|I|\rangle),$$

where Corr denotes the set of cohomological correspondences reviewed in §2.2 and $\boxtimes$ is the Hecke action, constructed in [Ari+22] and reviewed in [LW26, §4.3.2].

1.2.9. *Automorphic cohomological correspondence.* For local special correspondences $c_{V^i}^{\mathrm{loc}}$ of degrees d_i (defined in §2.3.4), the local-to-global construction of [LW26, Definition 4.41] produces a global cohomological correspondence

$$c_{V^I} \in \mathrm{Corr}_{\mathrm{Hk}_{G,I}^{\mathrm{glob}}, \mathrm{IC}_{V^I}}(\mathcal{P}_X, \mathcal{P}_X\langle d_I - 2|I|\rangle),$$

where $d_I = \sum_i d_i$.

1.2.10. *Assumptions.* We assume [LW26, Assumption 4.43] in [LW26, Setting 4.42], which is recalled in §2.4. In particular, we fix $x \in [r-1]$, and by assumption, there exists a unique element

$$c_{V^x \otimes V^{x+1}}^{\mathrm{loc}} \in \mathrm{Hom}^0(V^x \otimes V^{x+1} \langle -d_x - d_{x+1} + 2 \rangle, \mathrm{PL}_{X,h})$$

such that

$$c_{V^x \otimes V^{x+1}}^{\mathrm{loc}} \cdot \hbar = [c_{V^x}^{\mathrm{loc}}, c_{V^{x+1}}^{\mathrm{loc}}].$$

Let $[I] := [r-1]$. The local special correspondences $c_{V^i}^{\mathrm{loc}}$ for $i \neq x, x+1$ and $c_{V^x \otimes V^{x+1}}^{\mathrm{loc}}$ give the automorphic cohomological correspondence $c_{V^{[I]}}$ via the local-to-global construction in [LW26, Definition 4.41].

Let $V^{I,\mathrm{sw}}$ be obtained by interchanging the adjacent factors V^x, V^{x+1} .

1.3. **Main result.** Our main result is that [LW26, Conjecture 4.45] is true.

Main Theorem A (Automorphic commutator relation). *Under Assumption 2.1, we have that*

$$(A) \quad c_{V^I} - c_{V^{I,\mathrm{sw}}} = \Delta_{x,x+1,!}^{\mathrm{Hk}} c_{V^{[I]}}$$

in $\mathrm{Corr}_{\mathrm{Hk}_{G,I}^{\mathrm{glob}}, \mathrm{IC}_{V^I}}(\mathcal{P}_X, \mathcal{P}_X\langle d_I - 2|I|\rangle)$, where $\Delta_{x,x+1}^{\mathrm{Hk}}$ is the Hecke diagonal that merges the two legs.

1.4. **Idea of the proof: universal-local geometry.**

1.4.1. *Reduction to $|I| = 2$ and the semi-local case.* The main difficulty is already present for two legs, so we will assume $|I| = 2$ for now. Namely, we want to prove that $\epsilon^{\mathrm{glob}} := c_{V^I} - c_{V^{I,\mathrm{sw}}} - \Delta_{x,x+1,!}^{\mathrm{Hk}} c_{V^{[I]}}$ is zero. One important observation in [LW26] is that although the statement of the automorphic commutator relation is global, it suffices to prove its semi-local analogue $\epsilon^{\mathrm{semi-local}} = 0$ ([LW26, §4.6.3]), where semi-local refers to Beilinson–Drinfeld moduli spaces, i.e. local moduli spaces for a formal disk centered at a moving point on a fixed curve C (see [LW26, §4.3]).

1.4.2. *Original approach in [LW26].* The collision gives Cartesian squares

$$\begin{array}{ccccc} (LX/L^+G)_C & \xrightarrow{i} & (LX/L^+G)_{C^2} & \xleftarrow{j} & (LX/L^+G)_{C^2-\Delta_C} \\ \downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow{i} & C^2 & \xleftarrow{j} & C^2 - \Delta_C \end{array}$$

It is straightforward to see that $j^*\epsilon^{\text{semi-loc}} = 0$, and Assumption 2.1 on the local Plancherel algebra implies $i^*\epsilon^{\text{semi-loc}} = 0$. Excision expresses $\epsilon^{\text{semi-loc}}$ as a Gysin pushforward $\text{Gys}(\epsilon_0)$, and by the six-functor formalism, one obtains $c_1(T_C) \cdot \text{adj}(\epsilon_0) = 0$, where adj is an isomorphism. Therefore, if $c_1(T_C) \cdot$ is injective, we can conclude the proof. This is why additional assumptions were needed for the original approach in [LW26, §4.6.4].

1.4.3. *New approach in this article.* The local-global formalism via semi-local in [LW26] is not strong enough to investigate collisions and fusions.

We propose a fourth stage, “universal-local,” and use the concept of formal multidisks with two ordered marked sections to investigate fusion phenomena locally. We define universal-local versions of the objects, spaces, sheaves, and Hecke actions in [LW26, §4.3] and [BSV24].

Their collision geometry provides input for fusion arguments in sheaf theory and allows us to prove a universal-local commutator identity, using only the local assumption on the injectivity of $\hbar \cdot (-)$, without an additional global assumption on the injectivity of $c_1(T_C) \cdot (-)$.

Main Theorem B (Universal-local commutator identity). *Under Assumption 2.1 we have that*

$$(B) \quad c_{V,W}^{\text{univ-loc}} - c_{W,V}^{\text{univ-loc}} = \text{Gys}(c_{V \otimes W}^{\text{loc}}),$$

where $c_{V,W}^{\text{univ-loc}}$ and $c_{W,V}^{\text{univ-loc}}$ are defined by Proposition 5.2.

Then the six-functor formalism provides local-to-global compatibility and deduces Main Theorem A from Main Theorem B.

1.4.4. *Related ideas in literature.* Our definition of formal multidisks with two ordered marked sections is a special case of [Naf26, Definition 2.25]. In [LW26, Remark 4.49], the authors hoped but failed to prove the conjecture by reducing to the $[C = \mathbb{A}^1/\mathbb{G}_m]$ case. One could view Lemma 4.8 as a realization of this idea.

1.5. Organization of the paper. Section 2 fixes the local notation and recalls relevant settings of [LW26]. Section 3 defines formal multidisks with two marked sections and introduces classifying stack $\mathcal{M}_2$ of formal disks with marked sections, partial Hecke correspondences and the unit object analogous to those in the setting with one marked section. Section 4 studies the geometry of $\mathcal{M}_2$ and collisions. Section 5 studies the universal-local Hecke action and special correspondences. Section 6 states and proves the universal-local commutator identity. Section 7 completes the proof of Main Theorem A, by passing from universal-local to semi-local and from semi-local to global.

1.6. Generative AI usage. The proof of [LW26, Conjecture 4.45] was initially generated by the mathematical reasoning agent Rethlas [Ju+26], enhanced by OSVerify [Nie+26], an agentic verification framework. The author verified the proof, reorganized it into a paper and rewrote certain proofs for clearer exposition. Then the author used Codex to check typos and improve the writing. Details of generative AI usage are disclosed in Appendix A for transparency. The author is responsible for the correctness of the paper.

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2. NOTATIONS AND CONVENTIONS

We retain the conventions and setup of §1.2.

2.1. Sheaf theory. We use the sheaf theory on étale stacks of [LW26, §4.1], denoted by $\mathrm{Shv}(-)$.

2.2. Cohomological correspondences. For a correspondence $Y \xleftarrow{h_L} H \xrightarrow{h_R} Y$ of stacks, we define a cohomological correspondence c from $\mathcal{F}$ to $\mathcal{G}$ (supported on H with kernel K) to be an element

$$c \in \mathrm{Hom}^0(h_R^* \mathcal{F} \otimes K, h_L^! \mathcal{G}) \cong \mathrm{Hom}^0(h_{L,!}(h_R^* \mathcal{F} \otimes K), \mathcal{G}).$$

See [LW26, §4.2] for the functorialities of cohomological correspondences, including pullback and pushforward.

2.3. Geometric Setup. Now we recall the geometric setup in [LW26, §4.3].

2.3.1. Formal disk. Let $D = \mathrm{Spf} \mathbb{F}_q[[t]]$, and let $\mathrm{Aut}(D)$ denote the group of automorphisms preserving the origin.

2.3.2. Loop spaces. We put $Y_1 = LX/(L^+G \rtimes \mathrm{Aut}(D))$, $P_1 = L^+X/(L^+G \rtimes \mathrm{Aut}(D))$, where $LX(R) = X(R((t)))$, $L^+X(R) = X(R[[t]])$ are the *loop space* and *positive loop space*.

There is a natural map $i_1 : P_1 \hookrightarrow Y_1$, and the *unit object* $\delta_1 = i_{1,*}k_{P_1}$.

2.3.3. Local Hecke action. There is a natural action of $\mathrm{Sat}_{G,h} := \mathrm{Shv}(\mathrm{Hk}_G^{\mathrm{loc}})$ on $\mathrm{Sat}_{X,h} := \mathrm{Shv}(Y_1)$, where $\mathrm{Hk}_G^{\mathrm{loc}} := L^+G \backslash LG / (L^+G \rtimes \mathrm{Aut}(D))$ is the *loop-rotated local Hecke stack*.

2.3.4. Plancherel algebra. The *loop-rotated Plancherel algebra* $\mathrm{PL}_{X,h}$ is then defined to be

$$\mathrm{PL}_{X,h} = \underline{\mathrm{End}}_{\mathrm{Sat}_{G,h}}(\delta_1) \in \mathrm{Sat}_{G,h},$$

the internal endomorphism algebra of the unit object. In particular, we have that

$$(2.1) \quad \mathrm{Hom}_{\mathrm{Sat}_{G,h}}^0(\mathrm{IC}_V * \delta_1, \delta_1 \langle d \rangle) \simeq \mathrm{Hom}_{\check{G}}^0(V \langle -d \rangle, \mathrm{PL}_{X,h}).$$

Following [LW26, Definition 4.40], we define a *local special cohomological correspondence* of type $V \in \mathrm{Rep}(\check{G})^{\heartsuit, \omega}$ of degree d to be an element

$$c_V^{\mathrm{loc}} \in \mathrm{Hom}_{\check{G}}^0(V \langle -d \rangle, \mathrm{PL}_{X,h}).$$

Multiplication in $\mathrm{PL}_{X,h}$ corresponds to composition of local Hecke correspondences.

The parameter h is identified with the first Chern class of the tangent character of $\mathrm{Aut}(D)$, with the normalization of [LW26, §2.11].

2.4. Assumptions on Plancherel algebras. The automorphic commutator relation conjecture [LW26, Conjecture 4.45] requires two assumptions on Plancherel algebras ([LW26, Setting 4.42, Assumption 4.43]).

Assumption 2.1. *We assume the following.*

(1) *The map*

$$\begin{aligned} \hbar : \mathrm{Hom}_G^0(V \otimes W \langle -d_V - d_W + 2 \rangle, \mathrm{PL}_{X,\hbar}) \\ \rightarrow \mathrm{Hom}_G^0(V \otimes W \langle -d_V - d_W \rangle, \mathrm{PL}_{X,\hbar}) \end{aligned}$$

is injective.

(2) *There exists $c_{V \otimes W}^{\mathrm{loc}} \in \mathrm{Hom}^0(V \otimes W \langle -d_V - d_W + 2 \rangle, \mathrm{PL}_{X,\hbar})$, such that $\hbar c_{V \otimes W}^{\mathrm{loc}} = [c_V^{\mathrm{loc}}, c_W^{\mathrm{loc}}]$.*

Remark 2.2. When T^*X is a hyperspherical G -Hamiltonian space (in the sense of [BSV24, §3.5]), Assumption 2.1 is a direct consequence of the local relative Langlands conjecture [BSV24, Conjecture 8.5.2]. The flatness of $\mathrm{PL}_{X,\hbar}$ over $k[\hbar]$, and hence Assumption 2.1(1) in this case, is claimed in [26, Corollary 1.3].

3. MULTIDISKS AND MODULI SPACES

In this section, we define formal multidisks with two ordered marked sections and introduce analogues of the local Hecke stack, Hecke correspondences, and the unit object in this setting.

3.1. Formal multidisk. Let $S = \mathrm{Spec} R$ and $a, b \in R$. A formal multidisk (with two marked sections) is a formal S -scheme

$$D_{a,b,S} := \mathrm{Spf} \left(R[z]_{(z-a)(z-b)}^\wedge \right),$$

equipped with the ordered pair of sections s_a and s_b defined by $z = a$ and $z = b$.

3.2. Classifying stack of formal multidisks. Let $\mathcal{M}_2$ denote the stack for the étale topology whose objects over S are formal S -schemes with two ordered marked sections, étale locally isomorphic to some formal multidisk $D_{a,b,S}$. Morphisms are continuous isomorphisms over S preserving each marked section. Pullback is defined by completed base change.

Remark 3.1. The stack $\mathcal{M}_2$ is a multidisk analogue of the stack of formal disks with one marked section, namely

$$\mathcal{M}_1 \simeq B \mathrm{Aut}(D),$$

where $\mathrm{Aut}(D)$ is the group of automorphisms of the formal disk D preserving the origin, defined in §2.3.1.

We have the natural morphism

$$(3.1) \quad d_0 : \mathcal{M}_1 \hookrightarrow \mathcal{M}_2,$$

which repeats the marked section. By Proposition 4.1, d_0 is a closed immersion of finite presentation, and its complement is canonically identified with $\mathcal{M}_1 \times \mathcal{M}_1$.

3.3. Local Hecke stack for formal multidisks. For $(D; s_1, s_2) \in \mathcal{M}_2(S)$, write $D_i^\times$ for the punctured multidisk at s_i and $D^\times = D_1^\times \cap D_2^\times$. The local Hecke stack for multidisks $\mathrm{Hk}_2 \rightarrow \mathcal{M}_2$ has fiber

$$\mathrm{Hk}_2(D; s_1, s_2) = \{(\mathcal{E}_0, \mathcal{E}_2, \phi) \mid \phi : \mathcal{E}_0|_{D^\times} \xrightarrow{\sim} \mathcal{E}_2|_{D^\times}\},$$

where $\mathcal{E}_j$ are G -bundles on D . Define $\mathrm{Hk}_{i|2}$ similarly using $D_i^\times$ instead of $D^\times$.

3.4. Partial Hecke correspondences for formal multidisks. Let $\mathcal{B}_2$ be the stack whose S -points are

$$\mathcal{B}_2(S) = \{(D; s_1, s_2; \mathcal{E}) : (D; s_1, s_2) \in \mathcal{M}_2(S), \mathcal{E} \in \text{Bun}_G(D)\}$$

for a test scheme S . There is a natural projection $\mathcal{B}_2 \rightarrow \mathcal{M}_2$ forgetting the bundle.

Then we have Hecke correspondence in this setting

$$(3.2) \quad \begin{array}{ccc} & \text{Hk}_{i|2} & \\ h_L \swarrow & & \searrow h_R \\ \mathcal{B}_2 & & \mathcal{B}_2 \end{array}$$

where

$$\begin{aligned} h_R(D; s_1, s_2; \mathcal{E}, \mathcal{E}', \phi) &= (D; s_1, s_2; \mathcal{E}), \\ h_L(D; s_1, s_2; \mathcal{E}, \mathcal{E}', \phi) &= (D; s_1, s_2; \mathcal{E}'). \end{aligned}$$

3.5. Convolution Hecke stack for formal multidisks. For $(i, j) = (1, 2), (2, 1)$, define

$$\widetilde{\text{Hk}}_{ij} := \text{Hk}_{i|2} \times_{h_L, \mathcal{B}_2, h_R} \text{Hk}_{j|2}.$$

Over $(D; s_1, s_2)$, its objects are chains

$$\mathcal{E}_0 \xrightarrow{\phi_i} \mathcal{E}_1 \xrightarrow{\phi_j} \mathcal{E}_2$$

of G -bundles on D , where

$$\begin{aligned} \phi_i : \mathcal{E}_0|_{D_i^\times} &\xrightarrow{\sim} \mathcal{E}_1|_{D_i^\times}, \\ \phi_j : \mathcal{E}_1|_{D_j^\times} &\xrightarrow{\sim} \mathcal{E}_2|_{D_j^\times}. \end{aligned}$$

There are two convolution maps

$$\widetilde{\text{Hk}}_{12} \xrightarrow{m_{12}} \text{Hk}_2 \xleftarrow{m_{21}} \widetilde{\text{Hk}}_{21}$$

given by

$$m_{ij}(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2; \phi_i, \phi_j) = (\mathcal{E}_0, \mathcal{E}_2; \phi_j|_{D^\times} \circ \phi_i|_{D^\times}).$$

Remark 3.2. This is an analogue of the ordinary loop-rotated local Hecke stack Hk_G^{loc} for formal multidisks.

3.6. Unit object. Over $\mathcal{M}_2$, let Y_2 classify a G -bundle on the formal multidisk (D, s_a, s_b) and a section of its associated X -bundle on the punctured multidisk $D^\times$.

Let

$$i_2 : P_2 \hookrightarrow Y_2$$

be the substack which requires that the X -section extends to the whole multidisk.

Remark 3.3. The space $Y_2 \rightarrow \mathcal{M}_2$ is an analogue of $Y_1 := LX/(L^+G \rtimes \text{Aut}(D)) \rightarrow B\text{Aut}(D)$, and P_2 is an analogue of $P_1 := L^+X/(L^+G \rtimes \text{Aut}(D))$.

We define the unit object in this setting by

$$(3.3) \quad \delta_2 = i_{2,*}k_{P_2}.$$

3.7. Collision locus. The restrictions of (Y_2, P_2, δ_2) to the collision locus are Y_1, P_1, δ_1 of Section 2. We write

$$(3.4) \quad i : Y_1 \hookrightarrow Y_2, \quad i_P : P_1 \hookrightarrow P_2, \quad j : Y_{2,\text{disj}} \hookrightarrow Y_2.$$

Note that i, i_P are the base changes of $d_0 : \mathcal{M}_1 \hookrightarrow \mathcal{M}_2$. Therefore, i, i_P are closed immersions of finite presentation and j is a quasi-compact open immersion by Proposition 4.1.

4. GEOMETRY OF THE MODULI STACK OF FORMAL MULTIDISKS $\mathcal{M}_2$

In this section, we study the geometry of $\mathcal{M}_2$ and deduce its consequences for sheaf theory.

4.1. Geometry of $\mathcal{M}_2$.

4.1.1. Coordinate presentation and collision locus of $\mathcal{M}_2$.

Proposition 4.1. *We have the following description of $\mathcal{M}_2$ and the collision locus.*

- (1) *The morphism $\pi : \mathbb{A}_{\mathbb{F}_q}^2 \rightarrow \mathcal{M}_2$, defined by $(a, b) \mapsto (D_{a,b}, s_a, s_b)$, is a representable, surjective pro-smooth atlas. Its relation scheme $\mathbb{A}^2 \times_{\mathcal{M}_2} \mathbb{A}^2$ admits an inverse-limit presentation by affine schemes of finite presentation, smooth over $\mathbb{A}^2$ via either projection, with transition morphisms whose fibers are affine spaces.*
- (2) *The morphism $d_0 : \mathcal{M}_1 \hookrightarrow \mathcal{M}_2$ is a representable effective Cartier divisor of finite presentation. Its image is the locus where the two marked sections coincide, and its complement is canonically isomorphic to $\mathcal{M}_1 \times \mathcal{M}_1$.*

Proof. **Proof of (1).** We first prove the pro-smoothness of π .

Lemma 4.2. *We have the following Cartesian diagram*

$$\begin{array}{ccc} \mathrm{Spec} A & \xrightarrow{p} & S = \mathrm{Spec} R \\ \downarrow r & & \downarrow (D_{a,b}, s_a, s_b) \\ \mathbb{A}^2 & \xrightarrow{\pi} & \mathcal{M}_2 \end{array}$$

where $s = b - a$ and

$$(4.1) \quad A = R[a_0, b_0, a_1, b_1, a_2, b_2, \dots, (b_0(b_0 - sa_1)(b_0 + sa_1 + s^2b_1))^{-1}].$$

In particular, the maps p and π are pro-smooth.

Proof of Lemma 4.2. We denote the fiber product by $Z := \mathbb{A}^2 \times_{\mathcal{M}_2} S$.

We first describe Z and the map r explicitly. For an R -algebra R' , an R' -point of Z consists of a pair $(\alpha, \beta) \in \mathbb{A}^2(R')$ together with a specified marked isomorphism

$$\varphi : (D_{a,b}; s_a, s_b)_{R'} \xrightarrow{\sim} (D_{\alpha,\beta}; s_\alpha, s_\beta).$$

Translate the source coordinate by setting

$$t = z - a, \quad s = b - a, \quad f = t(t - s).$$

Let u denote the standard coordinate on $D_{\alpha,\beta}$. The isomorphism φ induces an isomorphism of complete R' -algebras

$$\varphi^\# : R'[u]_{(u-\alpha)(u-\beta)}^\wedge \xrightarrow{\sim} R'[t]_f^\wedge.$$

Define the polynomial

$$F(t) := \varphi^\#(u).$$

Since $\varphi^\#$ is continuous and R' -linear, it is determined by F . Preservation of the ordered marked sections requires

$$\alpha = F(0), \quad \beta = F(s).$$

Division by the monic polynomial f gives a unique expansion

$$F(t) = \sum_{n \geq 0} (a_n + b_n t) f^n.$$

Evaluating at the marked sections yields

$$\alpha = a_0, \quad \beta = a_0 + b_0 s.$$

By explicit coordinate calculation, one has that φ is a marked isomorphism, i.e. substitution by $u \mapsto F(t)$ gives a marked isomorphism, if and only if $b_0, b_0 - sa_1, b_0 + sa_1 + s^2b_1$ are invertible. Therefore, we have that

$$Z \simeq \operatorname{Spec} R[a_0, b_0, a_1, b_1, \dots, (b_0(b_0 - sa_1)(b_0 + sa_1 + s^2b_1))^{-1}],$$

and under this identification, the map $r : Z \rightarrow \mathbb{A}^2$ is given by

$$(a_n, b_n)_{n \geq 0} \mapsto (a_0, a_0 + b_0s).$$

Now we prove pro-smoothness of p and π .

For $N \geq 1$, set

$$Z_N = \operatorname{Spec} R[a_0, b_0, \dots, a_N, b_N, (b_0(b_0 - sa_1)(b_0 + sa_1 + s^2b_1))^{-1}].$$

Each Z_N is an open subscheme of $\mathbb{A}_R^{2N+2}$, hence is smooth and of finite presentation over R . Moreover,

$$Z_{N+1} \simeq Z_N \times_S \mathbb{A}_S^2, \quad Z \simeq \varprojlim_{N \geq 1} Z_N.$$

Thus the map p , i.e. the base change of π along a standard model $(D_{a,b}, s_a, s_b)$, is affine and pro-smooth.

For an arbitrary object $(\mathfrak{D}; s_1, s_2)$ over S , define

$$\mathcal{J} = \mathcal{I}_{s_1} \mathcal{I}_{s_2}, \quad \mathfrak{D}_N = V(\mathcal{J}^{N+1}).$$

The finite-level coordinate functor consists of functions v on $\mathfrak{D}_N$ for which substitution induces an isomorphism

$$\mathcal{O}_S[u] / ((u - s_1^*v)(u - s_2^*v))^{N+1} \xrightarrow{\sim} (\pi_N)_* \mathcal{O}_{\mathfrak{D}_N}, \quad u \mapsto v.$$

Here $\pi_N : \mathfrak{D}_N \rightarrow S$ is the structure morphism. This definition is intrinsic to the marked formal scheme, independent of the choice of coordinates. On standard charts it is represented by Z_N above, and marked isomorphisms identify these functors by pullback of functions. The resulting identifications satisfy the cocycle condition. Effective affine descent therefore produces the finite-level coordinate schemes Z_N over S . The inverse limit $\varprojlim_N Z_N$ represents $S \times_{\mathcal{M}_2} \mathbb{A}^2$ and is pro-smooth over S . Therefore, π is representable and pro-smooth. $\square$

It remains to prove the surjectivity of π and the stated properties of $\mathbb{A}^2 \times_{\mathcal{M}_2} \mathbb{A}^2$.

The map π is surjective by definition since every object of $\mathcal{M}_2$ is étale locally a standard model $(D_{a,b}, s_a, s_b)$ and hence π étale locally has a section.

Taking $S = \mathbb{A}^2 \xrightarrow{\pi} \mathcal{M}_2$ gives the relation scheme

$$\mathcal{R} := \mathbb{A}^2 \times_{\mathcal{M}_2} \mathbb{A}^2.$$

Apply Lemma 4.2 to $S = \mathbb{A}^2$, and we complete the proof of (1).

Now we prove (2).

Let $p_1, p_2 : \mathcal{R} \rightrightarrows \mathbb{A}^2$ denote the projections to the source and target presentations. On $\mathbb{A}^2$, the collision locus is defined by $b - a = 0$. On $\mathcal{R}$, the preceding calculation in the proof of Lemma 4.2 gives

$$p_2^*(b - a) = \beta - \alpha = F(s) - F(0) = b_0s = b_0p_1^*(b - a).$$

Since b_0 is invertible, the pulled-back ideals agree:

$$p_2^*((b - a)) = p_1^*((b - a)).$$

Their canonical identifications satisfy the cocycle condition, so the ideal $(b - a)$ descends along the faithfully flat atlas to an ideal sheaf on $\mathcal{M}_2$.

Since $(b - a)$ is an invertible ideal on $\mathbb{A}^2$, its descent gives an effective Cartier divisor on $\mathcal{M}_2$. The associated closed immersion is representable and of finite presentation by descent from $V(b - a) \hookrightarrow \mathbb{A}^2$.

Since completion along t^2 is the same as completion along t , this collision divisor is canonically identified with $\mathcal{M}_1$, compatibly with arbitrary base change.

On the complement, s is invertible. The Chinese remainder theorem then splits the completed algebra into the completions at $t = 0$ and $t = s$. Therefore, the complement is identified with $\mathcal{M}_1 \times \mathcal{M}_1$. $\square$

4.1.2. Universal formal disk. The universal pointed formal disk is the tautological formal family

$$\pi_{\text{univ}} : \mathfrak{D}_{\text{univ}} \longrightarrow \mathcal{M}_1,$$

together with the universal marked section

$$\sigma : \mathcal{M}_1 \longrightarrow \mathfrak{D}_{\text{univ}},$$

with $\pi_{\text{univ}} \circ \sigma = \text{id}_{\mathcal{M}_1}$.

For every scheme S and morphism $f : S \rightarrow \mathcal{M}_1$ classifying a pointed formal disk $(D_S \rightarrow S, p : S \rightarrow D_S)$, there is a canonical isomorphism of pointed formal families

$$(S \times_{\mathcal{M}_1} \mathfrak{D}_{\text{univ}}, \sigma_S) \simeq (D_S, p),$$

where σ_S is the base change of σ . These identifications are compatible with base change and isomorphisms of pointed formal disks.

The tangent line at the universal marked section is

$$\mathcal{T}_{\text{univ}} := \sigma^* T_{\mathfrak{D}_{\text{univ}}/\mathcal{M}_1}.$$

Its pullback along f is canonically identified with

$$f^* \mathcal{T}_{\text{univ}} \simeq p^* T_{D_S/S}.$$

4.1.3. The normal bundle of the collision locus.

Lemma 4.3. *The normal bundle of $d_0 : \mathcal{M}_1 \hookrightarrow \mathcal{M}_2$ is canonically isomorphic to $\mathcal{T}_{\text{univ}}$. Therefore, its first Chern class is $\hbar$, with the convention of [LW26, §2.11].*

Proof. Let $\mathcal{I}$ be the ideal of the collision divisor, and let $\sigma : \mathcal{M}_1 \rightarrow \mathfrak{D}_{\text{univ}}$ be as above. We first construct a canonical identification

$$(4.2) \quad \mathcal{I}/\mathcal{I}^2 \simeq \sigma^* \hat{\Omega}_{\mathfrak{D}_{\text{univ}}/\mathcal{M}_1}^1.$$

Here $\hat{\Omega}^1$ denotes continuous relative differentials.

On the coordinate atlas as in the proof of Lemma 4.2, write $t = z - a$ and $s = b - a$. The collision divisor is defined by $s = 0$, and its conormal line is generated by $\bar{s}$. The cotangent line at the common marked section is generated by $dt|_{t=0}$. Define the local identification by

$$\bar{s} \longmapsto dt|_{t=0}.$$

Under a marked coordinate change $u = F(t)$, the difference of the marked coordinates satisfies $s' = b_0 s$. On the collision locus, $F(t) = a_0 + b_0 t + t^2 H(t)$, and $F'(0) = b_0$. Hence $\bar{s}' = F'(0) \bar{s}$, and $du|_{t=0} = F'(0) dt|_{t=0}$. Therefore, the local identifications are compatible with marked coordinate changes and descend to the isomorphism (4.2). Taking duals, we get

$$N_{d_0} \simeq (\mathcal{I}/\mathcal{I}^2)^\vee \simeq \sigma^* T_{\mathfrak{D}_{\text{univ}}/\mathcal{M}_1}.$$

Under $\mathcal{M}_1 \simeq B \text{Aut}(D)$ and the convention of [LW26, §2.11], this tangent bundle is classified by $B\chi : B \text{Aut}(D) \rightarrow B\mathbb{G}_m$, and therefore, we have $N_{d_0} \simeq (B\chi)^* \mathcal{O}(-1)$, and

$$c_1(N_{d_0}) = (B\chi)^* c_1(\mathcal{O}(-1)) = \hbar.$$

□

The following lemma is straightforward by construction and definition.

Lemma 4.4. *There is a natural map given by the formal completion at two sections*

$$\gamma_C : C^2 \longrightarrow \mathcal{M}_2$$

for which the square

$$(4.3) \quad \begin{array}{ccc} C & \xrightarrow{\text{Coor}_C} & \mathcal{M}_1 \\ \Delta_C \downarrow & & \downarrow d_0 \\ C^2 & \xrightarrow{\gamma_C} & \mathcal{M}_2 \end{array}$$

is Cartesian, where Coor_C classifies the formal neighborhood of a point of C , equipped with its marked section. In particular,

$$\text{Coor}_C^* N_{d_0} \simeq N_{\Delta_C} \simeq T_C.$$

4.2. Purity.

Lemma 4.5 (Regular support and base change). *The inclusion $i_2 : P_2 \hookrightarrow Y_2$ is a representable closed immersion of finite presentation. The formation of the unit object $\delta_2 = i_{2,*}k_{P_2}$ is compatible with base change. Namely, for any morphism $g : B \rightarrow \mathcal{M}_2$, form the Cartesian squares*

$$\begin{array}{ccccc} P_B & \xrightarrow{i_B} & Y_B & \longrightarrow & B \\ \downarrow g_P & & \downarrow g_Y & & \downarrow g \\ P_2 & \xrightarrow{i_2} & Y_2 & \longrightarrow & \mathcal{M}_2 \end{array}$$

and then we have a canonical isomorphism

$$(4.4) \quad g_Y^* \delta_2 \xrightarrow{\sim} \delta_B := i_{B,*}k_{P_B}.$$

These isomorphisms are compatible with further base change.

Proof. The assertion about i_2 is proved *mutatis mutandis* as in the usual loop space case $i_1 : P_1 \hookrightarrow Y_1$ (see, e.g., [PR08, §1.a]).

The morphism i_2 is proper and hence proper base change gives

$$g_Y^* \delta_2 = g_Y^* i_{2,*}k_{P_2} \simeq i_{B,*}g_P^*k_{P_2} \simeq i_{B,*}k_{P_B} = \delta_B.$$

□

Remark 4.6. Take $B = \mathcal{M}_1$ and $g = d_0$, and we get $i^* \delta_2 \simeq \delta_1$, where $i : Y_1 \rightarrow Y_2$ is induced by d_0 .

Remark 4.7. Take $B = C^2$ and $g = \gamma_C$, and we get $\gamma_{C,Y}^* \delta_2 \simeq \delta_{C^2}$, where $\gamma_{C,Y} : Y_{C^2} \rightarrow Y_2$ is the induced projection.

Lemma 4.8 (Purity). *There are canonical isomorphisms*

$$(4.5) \quad i_P^! k_{P_2} \simeq k_{P_1} \langle -2 \rangle, \quad i^! \delta_2 \simeq \delta_1 \langle -2 \rangle$$

compatible with base change, where $i : Y_1 \hookrightarrow Y_2$ and $i_P : P_1 \hookrightarrow P_2$ are the inclusions of the collision loci.

Under these identifications, $i^* \text{Gys}(-)$ is identified with $\hbar \cdot (-)$, where for $\mathcal{F} \in \text{Shv}(Y_2)$, the Gysin pushforward is defined to be

$$(4.6) \quad \begin{aligned} \text{Gys} : \text{Hom}_{\text{Shv}(Y_1)}(i^* \mathcal{F}, \delta_1 \langle -2 \rangle) &\rightarrow \text{Hom}_{\text{Shv}(Y_2)}(\mathcal{F}, \delta_2), \\ (i^* \mathcal{F} \xrightarrow{c} \delta_1 \langle -2 \rangle) &\mapsto (\mathcal{F} \xrightarrow{\text{adj}} i_* i^* \mathcal{F} \xrightarrow{i_* c} i_* \delta_1 \langle -2 \rangle \xrightarrow{G_i} \delta_2), \end{aligned}$$

and $G_i : i_*\delta_1\langle -2 \rangle \cong i_*i^!\delta_2 \rightarrow \delta_2$ by adjunction.

Proof. If $X = \emptyset$, the assertions are immediate. Assume henceforth that $X \neq \emptyset$.

The Cartier divisor of Proposition 4.1 has a divisor-classifying map

$$(4.7) \quad \rho : \mathcal{M}_2 \longrightarrow [\mathbb{A}^1/\mathbb{G}_m],$$

and we have Cartesian squares

$$(4.8) \quad \begin{array}{ccccccc} P_1 & \xrightarrow{i_1} & Y_1 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & B\mathbb{G}_m \\ \downarrow i_P & & \downarrow i & & \downarrow d_0 & & \downarrow i_0 \\ P_2 & \xrightarrow{i_2} & Y_2 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

We want to apply locally pro-smooth base change theorem [LW26, Proposition 4.10] to these diagrams. Therefore, we need to verify the assumptions of the cited proposition, i.e. we want to find a pro-smooth atlas $\mathcal{J}(X) \rightarrow P_2$ such that the composition $\mathcal{J}(X) \rightarrow P_2 \rightarrow \mathcal{M}_2$ is pro-smooth.

(1) *Constructing a pro-smooth cover.* Define the $\mathbb{A}^2$ -scheme

$$\mathcal{J}(X)(R) = \{(a, b, v) : a, b \in R, v \in X(R[t]_{(t-a)(t-b)}^\wedge)\},$$

for a test scheme $S = \text{Spec} R \rightarrow \mathbb{A}^2$. Then $\mathcal{J}(X) \rightarrow \mathbb{A}^2$ is pro-smooth surjective. Similarly, $\mathcal{J}(G) \rightarrow \mathbb{A}^2$ is pro-smooth.

Trivializing the G -bundle gives

$$P_2 \times_{\mathcal{M}_2} \mathbb{A}^2 \simeq [\mathcal{J}(X)/\mathcal{J}(G)].$$

Then we have a representable pro-smooth surjection

$$u : \mathcal{J}(X) \rightarrow [\mathcal{J}(X)/\mathcal{J}(G)] \cong P_2 \times_{\mathcal{M}_2} \mathbb{A}^2 \rightarrow P_2.$$

Note that $\mathcal{J}(X)$ is an affine, hence quasi-compact quasi-separated, scheme. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^2 & \xrightarrow{\tilde{\rho}} & \mathbb{A}^1 \\ \downarrow \pi & & \downarrow \\ \mathcal{M}_2 & \xrightarrow{\rho} & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

where $\tilde{\rho}(a, b) = b - a$ is a smooth morphism. Therefore, we have a commutative diagram

$$(4.9) \quad \begin{array}{ccccccc} \mathcal{J}(X) & \longrightarrow & \mathbb{A}^2 & \xrightarrow{\tilde{\rho}} & \mathbb{A}^1 & & \\ \downarrow u & & \downarrow \pi & & \downarrow & & \\ & & P_1 & \xrightarrow{p_1} & \mathcal{M}_1 & \xrightarrow{\tilde{\rho}} & B\mathbb{G}_m \\ & \swarrow i_P & \downarrow & \swarrow d_0 & \downarrow & \swarrow i_0 & \\ P_2 & \xrightarrow{p_2} & \mathcal{M}_2 & \xrightarrow{\rho} & [\mathbb{A}^1/\mathbb{G}_m] & & \end{array}$$

where the two bottom squares are Cartesian. Thus the hypotheses of [LW26, Proposition 4.10] are satisfied.

(2) *Purity and proper base change.* Let

$$\alpha_z : k_{B\mathbb{G}_m}\langle -2 \rangle \xrightarrow{\sim} i_0^! k_{[\mathbb{A}^1/\mathbb{G}_m]}$$

be the zero-section purity isomorphism. Since i_0 is representable of finite presentation, [LW26, Proposition 4.10(1)] gives

$$(\tilde{\rho} \circ p_1)^* i_0^! k_{[\mathbb{A}^1/\mathbb{G}_m]} \xrightarrow{\sim} i_P^! (\rho \circ p_2)^* k_{[\mathbb{A}^1/\mathbb{G}_m]} \cong i_P^! k_{P_2}.$$

Apply α_z , and one proves the first formula in (4.5):

$$k_{P_1} \langle -2 \rangle \xrightarrow{\sim} i_P^! k_{P_2}.$$

The morphism i_2 is proper by Lemma 4.5. Extraordinary proper base change gives

$$i^! \delta_2 = i^! i_{2,*} k_{P_2} \simeq i_{1,*} i_P^! k_{P_2} \simeq \delta_1 \langle -2 \rangle.$$

This proves the second formula in (4.5).

(3) *Self-intersection.* Let $\mathcal{L}$ denote the universal line bundle on $B\mathbb{G}_m$. Then the self-intersection formula identifies the map

$$k_{B\mathbb{G}_m} \langle -2 \rangle \xrightarrow{\alpha_z} i_0^! k_{[\mathbb{A}^1/\mathbb{G}_m]} \longrightarrow i_0^* k_{[\mathbb{A}^1/\mathbb{G}_m]} \simeq k_{B\mathbb{G}_m}$$

with multiplication by $c_1(\mathcal{L})$. Therefore, by Lemma 4.3, one has that

$$(\tilde{\rho} \circ p_1)^* c_1(\mathcal{L}) = p_1^* c_1(N_{d_0}) = p_1^* \hbar.$$

□

Remark 4.9. This is the universal-local version of the calculation with $c_1(T_C)$ in [LW26, Lemma 4.56]. The locally pro-smooth base change theorem [LW26, Proposition 4.10] implies

$$(4.10) \quad \text{Coord}_C^* \hbar = c_1(T_C).$$

5. UNIVERSAL-LOCAL HECKE CORRESPONDENCES

5.1. Partial Hecke stacks and convolution. We have natural restriction maps for $i = 1, 2$:

$$(5.1) \quad \begin{aligned} \text{res}_i : \text{Hk}_{i|2} &\rightarrow \text{Hk}_G^{\text{loc}} \\ (D, s_1, s_2, \varphi : \mathcal{E}_0|_{D_i^\times} &\xrightarrow{\cong} \mathcal{E}_1|_{D_i^\times}) \mapsto (D, s_i, \varphi : \mathcal{E}_0|_{D_i^\times} \xrightarrow{\cong} \mathcal{E}_1|_{D_i^\times}) \end{aligned}$$

For $V \in \text{Rep}(\check{G})^{\vee, \omega}$, we pull back the Satake sheaf to obtain $\text{IC}_{V,i} := \text{res}_i^* \text{IC}_V$.

Recall the convolution Hecke stack from §3, the projection map $p_i : \widetilde{\text{Hk}}_{ij} \rightarrow \text{Hk}_{i|2}$ and the convolution map $m_{ij} : \widetilde{\text{Hk}}_{ij} \rightarrow \text{Hk}_2$. For sheaf $\mathcal{F}_i \in \text{Shv}(\text{Hk}_{i|2})$, we define

$$\mathcal{F}_i \star \mathcal{F}_j := m_{ij,!}(p_i^* \mathcal{F}_i \otimes p_j^* \mathcal{F}_j).$$

For $V, W \in \text{Rep}(\check{G})^{\vee, \omega}$, define

$$(5.2) \quad \begin{aligned} K_{V,W} &= \text{IC}_{V,1} \star \text{IC}_{W,2}, \\ K_{W,V} &= \text{IC}_{W,2} \star \text{IC}_{V,1}. \end{aligned}$$

5.2. Universal-local fusion.

Lemma 5.1 (Universal-local fusion comparison). *We have the following.*

- (1) *The canonical identification of $K_{V,W}$ and $K_{W,V}$ over the disjoint locus extends uniquely to an isomorphism $\eta_{V,W} : K_{V,W} \xrightarrow{\sim} K_{W,V}$ on Hk_2 .*
- (2) *Its restriction to the collision locus is the local Satake commutativity constraint.*
- (3) *For every smooth curve C , its pullback along $\gamma_C : C^2 \rightarrow \mathcal{M}_2$ is the usual fusion comparison.*

Proof. Proof of (1). Let $\pi : T = \mathbb{A}^2 \rightarrow \mathcal{M}_2$ be the coordinate presentation of Proposition 4.1, and put $T^\circ = T \setminus \Delta$. We construct the comparison over T and then descend it along π .

Fusion on the coordinate presentation. Put $\mathcal{H}_T = T \times_{\mathcal{M}_2} \text{Hk}_2$, and denote the pulled-back kernels by $K_{V,W,T}$ and $K_{W,V,T}$. Trivializing the input bundle gives the Beilinson–Drinfeld Grassmannian Gr_T and a morphism $e : \text{Gr}_T \rightarrow \mathcal{H}_T$. Set

$$\mathcal{A}_{V,W} = e^* K_{V,W,T}[2], \quad \mathcal{A}_{W,V} = e^* K_{W,V,T}[2].$$

Let $j : \mathrm{Gr}_{T^\circ} \hookrightarrow \mathrm{Gr}_T$ be the open immersion. Factorization identifies $j^* \mathcal{A}_{V,W} \cong j^* \mathcal{A}_{W,V} =: \mathcal{A}^\circ$. By [Zhu17, Proposition 5.4.2], we have a unique equivariant isomorphism

$$\mathcal{A}_{V,W} \simeq j_{!*} \mathcal{A}^\circ \simeq \mathcal{A}_{W,V},$$

which descends equivariantly along e and gives an isomorphism

$$\eta_T : K_{V,W,T} \xrightarrow{\sim} K_{W,V,T}.$$

Descent under coordinate changes. Put $R = T \times_{\mathcal{M}_2} T$, with projections q_1, q_2 . By abuse of notation, we also use q_1, q_2 to denote the base changed projections $\mathcal{H}_R := R \times_{\mathcal{M}_2} \mathrm{Hk}_2 \rightarrow \mathcal{H}_T$. The kernels have canonical descent isomorphisms

$$\theta_{V,W} : q_1^* K_{V,W,T} \xrightarrow{\sim} q_2^* K_{V,W,T}, \quad \theta_{W,V} : q_1^* K_{W,V,T} \xrightarrow{\sim} q_2^* K_{W,V,T},$$

since the restriction and convolution diagrams are intrinsic. To descend η_T , it suffices to prove

$$(5.3) \quad \theta_{W,V} \circ q_1^* \eta_T = q_2^* \eta_T \circ \theta_{V,W}.$$

Both sides agree over $R^\circ = q_1^{-1}(T^\circ) = q_2^{-1}(T^\circ)$. However, on bounded Satake supports, the kernels and their descent data descend to finite jet models. Then one again uses full faithfulness of intermediate extension to obtain (5.3) from its restriction to R° .

Proof of (2). Proper base change identifies the restrictions of $K_{V,W}$ and $K_{W,V}$ to the collision locus with $\mathrm{IC}_V \star \mathrm{IC}_W$ and $\mathrm{IC}_W \star \mathrm{IC}_V$, respectively. Under these identifications, the restriction of $\eta_{V,W}$ is the commutativity constraint obtained from fusion [Zhu17, Corollary 5.4.11].

Proof of (3). Part (3) is proved similarly. $\square$

5.3. Universal-local special correspondences.

5.3.1. Hecke stack with meromorphic X -sections. Let Hk_2^X be the Hecke stack with meromorphic X -sections. An object consists of a formal multidisk (D, s_1, s_2) , two G -bundles $\mathcal{E}_0, \mathcal{E}_1$ on D , an isomorphism

$$\varphi : \mathcal{E}_0|_{D^\times} \xrightarrow{\sim} \mathcal{E}_1|_{D^\times},$$

and sections $\xi_\nu \in \Gamma(D^\times, \mathcal{E}_\nu \times^G X)$, for $\nu = 0, 1$, satisfying $\varphi_* \xi_0 = \xi_1$. Define $\mathrm{Hk}_{i|2}^X$ similarly, requiring the modification φ to be defined on $D_i^\times$.

We have natural maps

$$\begin{array}{ccc} Y_2 & \xleftarrow{h_R} & \mathrm{Hk}_2^X & \xrightarrow{h_L} & Y_2 \\ & & \downarrow \pi & & \\ & & \mathrm{Hk}_2 & & \end{array}$$

where h_R, h_L remember $(\mathcal{E}_0, \xi_0)$ and $(\mathcal{E}_1, \xi_1)$, respectively, together with (D, s_1, s_2) , and the morphism π forgets the X -sections.

5.3.2. Hecke action. For $K \in \mathrm{Shv}(\mathrm{Hk}_2)$ with bounded Schubert support and $\mathcal{F} \in \mathrm{Shv}(Y_2)$, define the Hecke action by

$$(5.4) \quad K \boxtimes \mathcal{F} := h_{L,!}(\pi^* K \otimes h_R^* \mathcal{F}) \in \mathrm{Shv}(Y_2).$$

The pushforward is taken on the bounded support of $\pi^* K$. We also use $\boxtimes$ for the partial Hecke action defined by the analogous diagram for $\mathrm{Hk}_{i|2}^X$.

5.3.3. *Universal-local special correspondences.* Using $\eta_{V,W}$ in Lemma 5.1, we identify $K_{W,V}$ with $K_{V,W}$ and identify their induced actions on δ_2 .

Proposition 5.2 (Universal-local special correspondences). *The local special correspondences c_V^{loc}, c_W^{loc} canonically determine*

$$(5.5) \quad c_{V,W}^{univ-loc}, c_{W,V}^{univ-loc} \in \mathrm{Hom}_{Y_2}^0(K_{V,W} \boxtimes \delta_2, \delta_2 \langle d_V + d_W \rangle)$$

with the following properties.

- (1) *Their restrictions to the disjoint locus agree.*
- (2) *There is a canonical identification*

$$i^*(K_{V,W} \boxtimes \delta_2) \simeq \mathrm{IC}_{V \otimes W} * \delta_1.$$

Under this identification, $i^ c_{V,W}^{univ-loc}$ and $i^* c_{W,V}^{univ-loc}$ are identified with $c_V^{loc} c_W^{loc}$ and $c_W^{loc} c_V^{loc}$, respectively.*

- (3) *Pullback along $\gamma_C : C^2 \rightarrow \mathcal{M}_2$ gives the two semi-local special correspondences $c_{V \boxtimes W, \{1,2\}}^{semi-loc}$ and $c_{W \boxtimes V, \{1,2\}}^{semi-loc}$ defined in [LW26, equation (4.35)].*

Proof. We first extend a local correspondence to either marked section and then compose the resulting correspondences.

Step 1: a Cartesian square of regular loci. Fix $i \in \{1, 2\}$. Let $h_L, h_R : \mathrm{Hk}_{i|2}^X \rightarrow Y_2$ be the output and input maps as before and define $\mathcal{R}_i := P_2 \times_{Y_2, h_L} \mathrm{Hk}_{i|2}^X \times_{h_R, Y_2} P_2$, which parametrizes partial modifications for which both X -sections are regular.

We have the natural map $p_i : \mathcal{R}_i \rightarrow P_2$, which remembers the output bundle and its regular X -section. Define $p_{\mathrm{loc}} : \mathcal{R}_{\mathrm{loc}} \rightarrow P_1$ analogously for the usual local Hecke stack.

Restriction to the formal completion along the i -th section defines the horizontal maps in the square

$$(5.6) \quad \begin{array}{ccc} \mathcal{R}_i & \xrightarrow{q_i} & \mathcal{R}_{\mathrm{loc}} \\ p_i \downarrow & & \downarrow p_{\mathrm{loc}} \\ P_2 & \xrightarrow{r_i} & P_1. \end{array}$$

We claim that this square is Cartesian.

The claim is local on the base. In a standard coordinate presentation, write $A = R[t]_f^\wedge$, $f = t(t-s)$, as in the proof of Lemma 4.2 and let $\tau = t$ or $t-s$ define the selected section. For every $m \geq 1$, we have $A/(\tau^m) \simeq R[t]/(\tau^m) \simeq R[\tau]/(\tau^m)$, and hence $\hat{A}_{(\tau)} \simeq R[[\tau]]$.

Beauville–Laszlo gluing (see [Zhu17, Theorem 1.4.3 and proof of Corollary 1.4.5]) therefore identifies G -bundles over A with compatible G -bundles over $A[1/\tau]$ and $R[[\tau]]$, glued over $R((\tau))$. This equivalence also holds for bundles equipped with regular X -sections: using a G -equivariant closed embedding of X into a representation (assumed in §1.2), one glues the associated vector-bundle sections and imposes the defining equations of X . Full faithfulness and functoriality of gluing show that (5.6) is Cartesian, compatibly with base change.

Step 2: extension of a local correspondence. Fix a representation U , and let K_{loc} and K_i be the pullbacks of the Satake sheaf IC_U to $\mathcal{R}_{\mathrm{loc}}$ and $\mathcal{R}_i$, respectively. By construction,

$$K_i \simeq q_i^* K_{\mathrm{loc}}.$$

We work throughout on the Satake support of U . On this support, p_{loc} is proper of finite presentation, and hence p_i is proper of finite presentation by (5.6).

By adjunction and proper base change, we have that for every $n \in \mathbb{Z}$,

$$(5.7) \quad \begin{aligned} \mathrm{Hom}_{Y_1}^0(\mathrm{IC}_U * \delta_1, \delta_1 \langle n \rangle) &\simeq \mathrm{Hom}_{P_1}^0(p_{\mathrm{loc},!} K_{\mathrm{loc}}, k_{P_1} \langle n \rangle), \\ \mathrm{Hom}_{Y_2}^0(\mathrm{IC}_{U,i} \boxtimes \delta_2, \delta_2 \langle n \rangle) &\simeq \mathrm{Hom}_{P_2}^0(p_{i,!} K_i, k_{P_2} \langle n \rangle). \end{aligned}$$

Proper base change in (5.6) gives

$$(5.8) \quad p_{i,!} K_i \simeq r_i^* p_{\mathrm{loc},!} K_{\mathrm{loc}}.$$

Given a local correspondence

$$c : \mathrm{IC}_U * \delta_1 \longrightarrow \delta_1 \langle n \rangle,$$

let $\bar{c} : p_{\mathrm{loc},!} K_{\mathrm{loc}} \rightarrow k_{P_1} \langle n \rangle$ be its image under the first identification in (5.7). Pulling back $\bar{c}$ defines

$$p_{i,!} K_i \xrightarrow{\sim} r_i^* p_{\mathrm{loc},!} K_{\mathrm{loc}} \xrightarrow{r_i^* \bar{c}} k_{P_2} \langle n \rangle.$$

Under the second identification in (5.7), this corresponds to a canonical morphism

$$c_{U,i} : \mathrm{IC}_{U,i} \boxtimes \delta_2 \rightarrow \delta_2 \langle n \rangle.$$

Step 3: Construction of $c_{V,W}^{\mathrm{univ-loc}}$ and $c_{W,V}^{\mathrm{univ-loc}}$. Apply Step 2 to c_V^{loc} at leg 1 and c_W^{loc} at leg 2 to obtain $c_{V,1} : \mathrm{IC}_{V,1} \boxtimes \delta_2 \rightarrow \delta_2 \langle d_V \rangle$ and $c_{W,2} : \mathrm{IC}_{W,2} \boxtimes \delta_2 \rightarrow \delta_2 \langle d_W \rangle$. By proper base change and the projection formula, we have that

$$K_{V,W} \boxtimes \mathcal{F} \cong \mathrm{IC}_{W,2} \boxtimes (\mathrm{IC}_{V,1} \boxtimes \mathcal{F})$$

and

$$K_{W,V} \boxtimes \mathcal{F} \cong \mathrm{IC}_{V,1} \boxtimes (\mathrm{IC}_{W,2} \boxtimes \mathcal{F}).$$

Identifying $K_{V,W} \cong K_{W,V}$ by $\eta_{V,W}$, we obtain the morphisms $c_{V,W}^{\mathrm{univ-loc}}, c_{W,V}^{\mathrm{univ-loc}}$ by composing $c_{V,1}$ and $c_{W,2}$ in the two orders.

Then (1) is straightforward by our construction.

On the collision locus, completion along either marked section is the entire one-point formal disk. Part (2) follows by construction and proper base change.

Finally, restriction to the selected formal completion, gluing, and the base-change isomorphisms above are compatible with pullback along γ_C . Therefore, under (5.7), the restriction and extension operations of [LW26, equations (4.29)–(4.30)] are precisely the pullback of $\bar{c}$ and the inverse adjunction used in Step 2. Thus the single-leg correspondences specialize to those of [LW26, §4.6]. Their ordered compositions are the semi-local correspondences of [LW26, equation (4.35)], proving (3). $\square$

6. THE UNIVERSAL-LOCAL COMMUTATOR IDENTITY

In this section, we state and prove Main Theorem B.

6.1. Gysin map. Recall that, using the identification $i^*(K_{V,W} \boxtimes \delta_2) \simeq \mathrm{IC}_{V \otimes W} * \delta_1$, purity gives the Gysin map

$$(6.1) \quad \begin{aligned} \mathrm{Gys} : \mathrm{Hom}_{Y_1}^0(\mathrm{IC}_{V \otimes W} * \delta_1, \delta_1 \langle d_V + d_W - 2 \rangle) \\ \longrightarrow \mathrm{Hom}_{Y_2}^0(K_{V,W} \boxtimes \delta_2, \delta_2 \langle d_V + d_W \rangle), \\ u \longmapsto \left(K_{V,W} \boxtimes \delta_2 \longrightarrow i_* i^*(K_{V,W} \boxtimes \delta_2) \right. \\ \left. \xrightarrow{i_* u} i_* i^! \delta_2 \langle d_V + d_W \rangle \longrightarrow \delta_2 \langle d_V + d_W \rangle \right). \end{aligned}$$

By Lemma 4.8, we have that

$$(6.2) \quad i^* \mathrm{Gys}(u) = \hbar u.$$

Main Theorem B (Universal-local commutator identity). *Under Assumption 2.1 we have that*

$$(6.3) \quad c_{V,W}^{\text{univ-loc}} - c_{W,V}^{\text{univ-loc}} = \text{Gys}(c_{V \otimes W}^{\text{loc}}),$$

where $c_{V,W}^{\text{univ-loc}}$ and $c_{W,V}^{\text{univ-loc}}$ are defined by Proposition 5.2.

Proof. Define $\epsilon^{\text{univ-loc}} = c_{V,W}^{\text{univ-loc}} - c_{W,V}^{\text{univ-loc}} - \text{Gys}(c_{V \otimes W}^{\text{loc}})$. By Proposition 5.2 and the support of the Gysin term, $j^* \epsilon^{\text{univ-loc}} = 0$. By Assumption 2.1(2), Proposition 5.2, and (6.2),

$$(6.4) \quad i^* \epsilon^{\text{univ-loc}} = [c_V^{\text{loc}}, c_W^{\text{loc}}] - \hbar c_{V \otimes W}^{\text{loc}} = 0.$$

We have the excision triangle

$$j!j^*(K_{V,W} \boxtimes \delta_2) \longrightarrow K_{V,W} \boxtimes \delta_2 \longrightarrow i_*i^*(K_{V,W} \boxtimes \delta_2)$$

by [LW26, Proposition 4.12]. Apply $\text{Hom}(-, \delta_2 \langle d_V + d_W \rangle)$, by exactness on degree-zero morphisms and $j^* \epsilon^{\text{univ-loc}} = 0$, one obtains

$$(6.5) \quad \epsilon_0 \in \text{Hom}_{Y_1}^0(\text{IC}_{V \otimes W} * \delta_1, \delta_1 \langle d_V + d_W - 2 \rangle)$$

such that

$$\epsilon^{\text{univ-loc}} = \text{Gys}(\epsilon_0).$$

Equations (6.2) and (6.4) imply $\hbar \epsilon_0 = 0$. By Assumption 2.1(1), one has $\epsilon_0 = 0$, and hence $\epsilon^{\text{univ-loc}} = 0$. $\square$

7. PROOF OF MAIN THEOREM A

In this section, we prove Main Theorem A.

7.1. Reduction to the case $|I| = 2$. The reduction from the general case to the two-leg case is carried out in [LW26, §4.6.1], without additional assumptions. It follows from the functorialities of cohomological correspondences: compatibility of pushforward with composition and of pullback with composition.

7.2. From universal-local to semi-local. Fix a smooth curve C . For $j = 1, 2$, define

$$Y_{C^j} := C^j \times_{\mathcal{M}_j} Y_j, \quad P_{C^j} := C^j \times_{\mathcal{M}_j} P_j,$$

using $\text{Coor}_C : C \rightarrow \mathcal{M}_1$ for $j = 1$ and $\gamma_C : C^2 \rightarrow \mathcal{M}_2$ for $j = 2$. These are the semi-local moduli of meromorphic and regular X -sections, respectively. Write

$$g_j : Y_{C^j} \rightarrow Y_j, \quad g_{P,j} : P_{C^j} \rightarrow P_j$$

for the projections. Let $i_{j,C} : P_{C^j} \hookrightarrow Y_{C^j}$ be the base change of i_j , and set $\delta_{C^j} := i_{j,C,*} k_{P_{C^j}}$. Lemma 4.5 gives canonical identifications

$$g_j^* \delta_j \simeq \delta_{C^j}.$$

The Cartesian square (4.3) induces a Cartesian square

$$(7.1) \quad \begin{array}{ccc} Y_C & \xrightarrow{g_1} & Y_1 \\ \Delta_Y \downarrow & & \downarrow i \\ Y_{C^2} & \xrightarrow{g_2} & Y_2, \end{array}$$

where Δ_Y is induced by the diagonal $\Delta_C : C \hookrightarrow C^2$. In particular, proper base change gives

$$(7.2) \quad g_2^* i_* \simeq \Delta_{Y,*} g_1^*.$$

The Hecke correspondences also commute with this base change. Put

$$\mathrm{Hk}_{2,C}^X := C^2 \times_{\mathcal{M}_2} \mathrm{Hk}_2^X.$$

Both squares in

$$(7.3) \quad \begin{array}{ccccc} Y_{C^2} & \xleftarrow{h_{R,C}} & \mathrm{Hk}_{2,C}^X & \xrightarrow{h_{L,C}} & Y_{C^2} \\ \downarrow g_2 & & \downarrow g_{\mathrm{Hk}} & & \downarrow g_2 \\ Y_2 & \xleftarrow{h_R} & \mathrm{Hk}_2^X & \xrightarrow{h_L} & Y_2 \end{array}$$

are Cartesian; the analogous statement holds for the one-point and ordered convolution diagrams. All Hecke pushforwards below are taken on the bounded Satake supports. Let $K_{V,W,C}$ be the pullback of $K_{V,W}$ to $\mathrm{Hk}_{2,C} := C^2 \times_{\mathcal{M}_2} \mathrm{Hk}_2$, and put

$$\mathcal{F} := K_{V,W} \boxtimes \delta_2, \quad \mathcal{F}_C := K_{V,W,C} \boxtimes \delta_{C^2}.$$

Proper base change in (7.3) identifies $g_2^* \mathcal{F} \simeq \mathcal{F}_C$. By Lemmas 5.1 and 5.2, the pullbacks of the two ordered universal-local correspondences are the two semi-local correspondences.

It remains to check the Gysin term. Put $d = d_V + d_W$, and denote the purity isomorphisms by

$$\alpha_i : \delta_1 \langle -2 \rangle \xrightarrow{\sim} i^! \delta_2, \quad \alpha_C : \delta_C \langle -2 \rangle \xrightarrow{\sim} \Delta_Y^! \delta_{C^2}.$$

Their compatibility is by Lemma 4.8. More explicitly, let $i_0 : B\mathbb{G}_m \hookrightarrow [\mathbb{A}^1/\mathbb{G}_m]$ be the zero section. The composite $\rho \circ \gamma_C$ classifies the diagonal Δ_C , by (4.3). Writing $p_2 : P_2 \rightarrow \mathcal{M}_2$ for the structural morphism, we therefore have a diagram with both squares Cartesian:

$$\begin{array}{ccccc} P_C & \xrightarrow{g_{P,1}} & P_1 & \longrightarrow & B\mathbb{G}_m \\ \Delta_P \downarrow & & \downarrow i_P & & \downarrow i_0 \\ P_{C^2} & \xrightarrow{g_{P,2}} & P_2 & \xrightarrow{\rho \circ p_2} & [\mathbb{A}^1/\mathbb{G}_m]. \end{array}$$

Here Δ_P is induced by Δ_C .

Their compatibility gives a commutative diagram

$$(7.4) \quad \begin{array}{ccc} g_1^* \delta_1 \langle -2 \rangle & \xrightarrow{g_1^* \alpha_i} & g_1^* i^! \delta_2 \\ \sim \downarrow & & \sim \downarrow (7.1) \\ \delta_C \langle -2 \rangle & \xrightarrow{\alpha_C} & \Delta_Y^! \delta_{C^2}. \end{array}$$

Let

$$\epsilon_i : i_* \delta_1 \langle -2 \rangle \xrightarrow{i_* \alpha_i} i_* i^! \delta_2 \xrightarrow{\text{counit}} \delta_2$$

be the Gysin morphism, and define $\epsilon_C : \Delta_{Y,*} \delta_C \langle -2 \rangle \rightarrow \delta_{C^2}$ in the same way using α_C . By (7.2), (7.4), and compatibility of the exchange morphism with the counit, $g_2^* \epsilon_i$ identifies with ϵ_C .

Write $\mathrm{Gys}_{\mathrm{univ-local}}$ for the Gysin operation occurring in (6.3). For a morphism

$$a : i^* \mathcal{F} \longrightarrow \delta_1 \langle d-2 \rangle,$$

its definition in (6.1) is the composite

$$\begin{aligned} \mathcal{F} &\xrightarrow{\text{unit}} i_* i^* \mathcal{F} \\ &\xrightarrow{i_* a} i_* \delta_1 \langle d-2 \rangle \\ &\xrightarrow{\epsilon_i \langle d \rangle} \delta_2 \langle d \rangle. \end{aligned}$$

Pulling back by g_2 , the preceding identifications transform this composite into

$$\begin{aligned} \mathcal{F}_C &\xrightarrow{\text{unit}} \Delta_{Y,*} \Delta_Y^* \mathcal{F}_C \\ &\xrightarrow{\Delta_{Y,*}(g_1^* a)} \Delta_{Y,*} \delta_C \langle d-2 \rangle \\ &\xrightarrow{\epsilon_C \langle d \rangle} \delta_{C^2} \langle d \rangle. \end{aligned}$$

This is precisely the definition of $\text{Gys}_{\text{semi-local}}(g_1^* a)$ in [LW26, equation (4.56)]. We have therefore proved

$$(7.5) \quad g_2^* \text{Gys}_{\text{univ-local}}(a) = \text{Gys}_{\text{semi-local}}(g_1^* a).$$

Under the collision identification $i^* \mathcal{F} \simeq \text{IC}_{V \otimes W} * \delta_1$, take $a = c_{V \otimes W}^{\text{loc}}$. Its pullback $g_1^* a$ is $c_{V \otimes W, \{1\}}^{\text{semi-loc}}$. Apply g_2^* to (6.3) and use Lemma 5.2 and (7.5), and one has

$$(7.6) \quad c_{V \boxtimes W, \{1,2\}}^{\text{semi-loc}} - c_{W \boxtimes V, \{1,2\}}^{\text{semi-loc}} = \text{Gys}_{\text{semi-local}}(c_{V \otimes W, \{1\}}^{\text{semi-loc}}).$$

This proves [LW26, Conjecture 4.55] for C .

7.3. From semi-local to global. The deduction of Main Theorem A (the global automorphic commutator relation, [LW26, Conjecture 4.45]) from the semi-local commutator relation (7.6) ([LW26, Conjecture 4.55]) is given in [LW26, §4.6.2, §4.6.3] without additional assumptions, by applying the functorialities of cohomological correspondences and unwinding the definitions (see Definitions 4.40 and 4.41 and equations (4.49), (4.62) and (4.67) in [LW26]).

Therefore, this completes the proof of Main Theorem A.

APPENDIX A. DETAILS OF GENERATIVE AI USAGE

Generative AI contributed to the mathematical argument and the preparation of this article. The proof was initially produced by the agentic system Rethlas [Ju+26] in about 1 hour, enhanced by the verification system OSVerify [Nie+26]. Then the author verified the draft and rewrote it into a paper. Subsequent generative AI assistance was used to revise the L^AT_EX presentation and improve the writing.

By contrast, Danus (running on GPT-6-Astra) [Liu+26] failed to solve it within 5 hours, and the GPT-6-Astra Max web interface failed to solve it in 3 attempts with the same prompt given in §A.1.

For transparency, this appendix records the details of AI usage in proof generation and verification.

A.1. Problem prompt.

Prove Conjecture 4.45 Automorphic commutator relation in without Assumption 4.46, of arXiv 2504.00275.

The logs of the generation agent explicitly interpret this task as retaining [LW26, Setting 4.42].

A.2. Experiment settings. The experiment took place on September 22, 2026. The model identifiers and reasoning-effort settings are listed below. Generation and review used the Codex CLI, version 0.155.1, with live web search enabled. The agent system Rethlas [Ju+26] was adapted for OSVerify [Nie+26], and the review backend was configured as a decomposing verifier.

Recorded item	Value
Generator	<code>gpt-6-astra</code> , reasoning effort <code>max</code>
Reviewer	<code>gpt-6-astra</code> , reasoning effort <code>xhigh</code>
Generation agent	active time ≈ 36 min
Verification agent	active time ≈ 25 min
Total elapsed time	active time ≈ 61 min

A.3. Reasoning trajectory. The reasoning trajectory is reconstructed from retained memory entries and log files. We summarize it in Figure 1.

- 1. Initial reduction.** The initial analysis retained [LW26, Assumption 4.43] and reduced the problem to the case of two legs.
- 2. Identifying the limitations of the original approach in [LW26].** It identified a gap in the naive argument that a correspondence is determined by its restrictions to the diagonal and its complement. As a test case, the record uses the nonzero diagonal class on the square of an elliptic curve: it restricts to zero on the complement, while its self-intersection is $c_1(T_C) = 0$. A second test, with trivial G and X a point, showed why local $\hbar$ -injectivity need not survive specialization to a curve. These tests invalidated the proposed proof methods and helped the agent understand why [LW26, Assumption 4.46] was needed in the original approach of [LW26].
- 3. Investigating two approaches.** Two internal branches investigated “universal collision” and “factorization residues.” The first developed a space of strictly marked formal multidisks. The second investigated whether a factorization residue could supply a canonical divided commutator. The record did not establish the comparison needed to apply the higher-algebra results, and that direct route was set aside.
- 4. Constructing the universal collision.** The selected approach constructed an explicit coordinate presentation for $\mathcal{M}_2$, identified its collision divisor with $B \operatorname{Aut}(D)$, and computed the normal line as the tangent character. The resulting self-intersection class is $\hbar$. The same

Rethlas' Exploration Trajectory

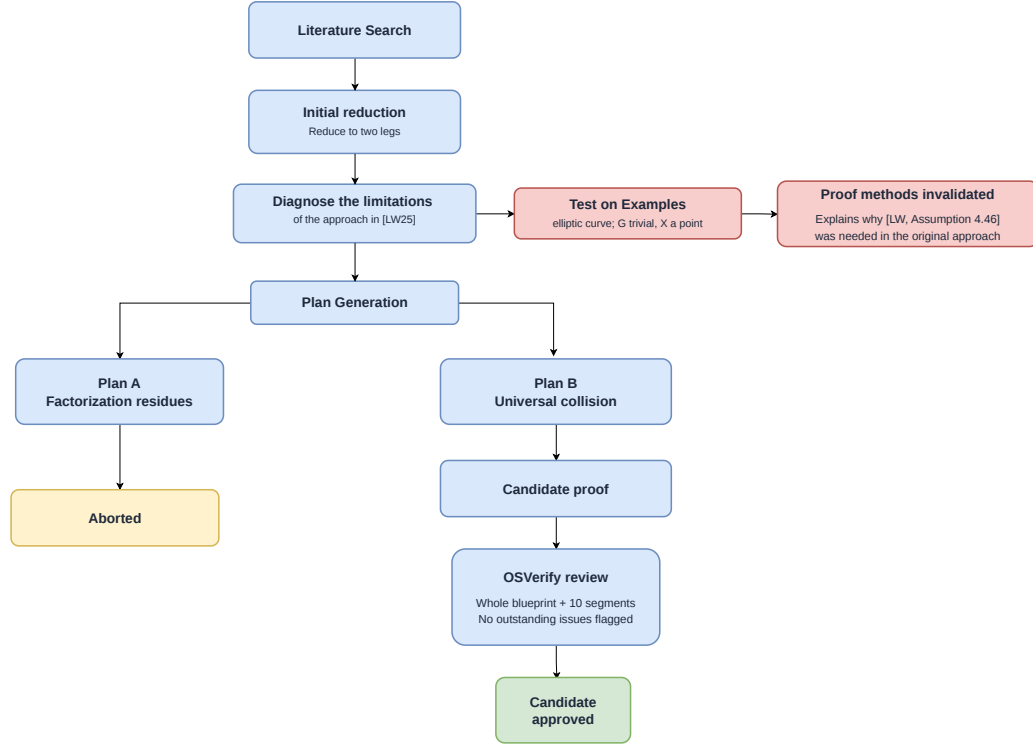


FIGURE 1. Rethlas–OSVerify reasoning trajectory

branch developed the regular-section condition and the pro-smooth presentations used for purity.

5. **Constructing the morphisms and their descent.** The agent separated descent of the Satake fusion comparison from construction of the local correspondence maps. The former uses intermediate extension at finite coordinate stages. The latter uses Beauville–Laszlo patching and proper base change. They avoid inferring equivalence of derived morphisms solely from equality on classical coordinate charts.
6. **Assembling the proof.** The final blueprint applies excision to the universal error term, writes it as $\mathrm{Gys}(u)$, and uses self-intersection to obtain $\hbar u = 0$ and hence $u = 0$. Only after this step does the proof specialize to a curve, pass to the period sheaf, and compose with the remaining legs.
7. **OSVerify finally approved the candidate solution.** OSVerify reviewed the submitted blueprint as a whole and in ten additional segments, and did not flag outstanding issues.

Then the author verified the generated proof, reorganized it into a paper and rewrote certain proofs for clearer exposition.

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